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On fermionic T-duality of sigma modes on AdS backgrounds

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Abstract: We study the fermionic T-duality symmetry of integrable Green-Schwarz sigma models on AdS backgrounds. We show that the sigma model on $AdS_5 \times S^1$ background is self-dual under fermionic T-duality. We also construct new integrable sigma models on $AdS_2 \times CP^n$. These backgrounds could be realized as supercosets of SU supergroups for arbitrary n, but could also be realized as supercosets of OSp supergroups for $n = 1, 3$. We find that the supercosets based on SU supergroups are self-dual under fermionic T-duality, while the supercosets based on OSp supergroups are not. However, the reasons of OSp supercosets being not self-dual under fermionic T-duality are different. For $OSp(6|2)$ case, corresponding to $AdS_2 \times CP^3$ background, the failure is due to the singular fermionic quadratic terms, just like $AdS_4 \times CP^3$ case. For $OSp(3|2)$ case, the failure is due to the shortage of right number of κ-symmetry to gauge away the fermionic degrees of freedom, even though the fermionic quadratic term is not singular any more. More general, for the supercosets of the OSp supergroups with superalgebra $B(n, m)$, including $AdS_2 \times S^{2n}$ and $AdS_4 \times S^{2n}$ backgrounds, the sigma models are not self-dual under fermionic T-duality as well, obstructed by the κ-symmetry.

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JHEP12(2009)051

Contents

1 Introduction

Recently, it was found that $\mathcal{N} = 4$ SYM gluon scattering amplitudes display a non-trivial symmetry called the dual conformal invariance [1–3], originating from self-dual symmetry of the AdS_5 under T-duality. This dual conformal symmetry can be extended to the full dual superconformal symmetry [4, 6], considering full set of supergluon amplitudes. In this case, the existence of fermionic T-duality transformation play a key role. The $AdS_5 \times S^5$ Green-Schwarz superstring theory is self-dual under a combination of bosonic and fermionic T-duality. This fact explains the superconformal and the dual superconformal invariance in $\mathcal{N} = 4$ SYM.

It turns out that the T-duality is closely related to the integrability of the sigma models [5, 6]. For AdS_n background, the sigma models are self-dual under bosonic T-duality, suggesting the local Noether charges of dual model are related to the non-local charges of the original model and vice versa. This relation could be generalized to the integrable super-coset models which are self-dual under the combination of bosonic and fermionic T-duality transformations. In fact, both bosonic and fermionic T-duality could be understood as the discrete automorphism of the global symmetry algebra.

It is interesting to investigate if and under what conditions other integrable sigma models could be self-dual under fermionic T-duality. In [7], the authors considered more general integrable Green-Schwarz sigma models on AdS backgrounds. They showed that the sigma models on $AdS_p \times S^p (p = 2, 3)$ background which are supercosets of PSU supergroups are self-dual under fermionic T-duality, while the non-critical AdS_2 and AdS_4 models and the critical $AdS_4 \times CP^3$ which all are supercosets of OSp supergroups are not.

JHEP12(2009)051

They also argued that in general the models which are supercosets of ortho-symplectic groups are not self-dual under fermionc T-duality, hence are short of the dual superconformal symmetry even its dual models exist. It was argued that the absence of fermionic self-duality in OSp modes is due to the lack of appropriate fermionic quadratic terms, because the Cartan-Killing bilinear form of OSp group is only nonvanishing for the products of different fermionic generators.

In this paper, we explore this problem further by analyzing other integrable Green-Schwarz sigma models on AdS backgrounds. We consider integrable supercosets with Z_4 grading. The existence of Z_4 grading of supercosets allows us to construct one-parameter families of flat currents [8, 9], which in turn allow for the construction of infinitely many non-local charges [12]. We first show that the sigma model with $AdS_5 \times S^1$ background, which is the supercoset of SU$(2,2|2)$ supergroup, is self-dual under fermionic T-duality. We then present a series of new integrable Green-Schwarz sigma models with the backgrounds $AdS_2 \times CP^n$. Considering the critical dimension of superstring, we only focus on the cases with $n \leq 4$. These backgrounds could be taken as supercosets of SU supergroups for arbitrary n. However, for $n = 1, 3$ the backgrounds could also be realized as supercosets of OSp supergroups. We show explicitly that all of the SU cases are self-dual under fermionic T-duality, while the OSp cases are not. Our study on $AdS_2 \times CP^n$ with $n = 1, 3$ shows that even the bosonic background is the same, the different supersymmetrizations may have different behavior under fermionic T-duality.

Moreover, we find that in the $n = 1$ Osp supergroup case, corresponding to $AdS_2 \times CP^1$ background, even though the sigma model has regular fermionic quadratic term, it fails to be self-dual under fermionic T-duality. The failure is due to the shortage of κ-symmetry to gauge away the right number of fermionic degrees of freedom. This happens for other backgrounds, including $AdS_2 \times S^{2n}$ and $AdS_4 \times S^{2n}$. Therefore, in general, the sigma models on supercosets of OSp supergroup can not be self-dual under fermionic T-duality, but due to different reasons. For OSp supergroups with superalgebra of types $C(n)$ and $D(m, n)$, the failure stems from the singular fermionic quadratic terms, while for OSp supergroups with superalgebra of type $B(m, n)$, the failure comes from the shortage of κ-symmetry.

This paper is organized as follows. In section 2 we show that the $AdS_5 \times S^1$ background is self-dual under a combination of bosonic and fermionic T-duality. In section 3, we study the $AdS_2 \times CP^n$ cases. We first discuss the SU cases. After presenting their superalgebra and Z_4 automorphism which are crucial for the integrability, we preform the T-duality via Buscher's procedure, and show that the supercoset models are self-dual under T-duality. Then we turn to the OSp cases, and show that they are not self-dual, due to different reasons. In section 4, we conclude and present a brief discussion. We collect some technical details into the appendices. In appendix A, we give the definition of the generators of the superalgebra SU$(1, 1|n)$. And in appendix B, we discuss the κ-symmetry in $AdS_{2n} \times S^{2m}$ backgrounds.

2 $AdS_5 \times S^1$ background

In this section we consider the Green-Schwarz sigma-model on $AdS_5 \times S^1$ using the supercoset manifold SU$(2, 2|2)/($SO$(4, 1) \times$ SO$(3))$. It was first pointed out by Polyakov [10] that noncritical $AdS_p \times S^q$ are conformal invariant and should be dual to gauge theories with less or no supersymmetries. And later on in [11], Klebanov and Maldacena found the $AdS_5 \times S^1$ solution in the low energy supergravity effective action of six dimensional noncritical string theory with Ramond-Ramond flux and in the presence of space-time filling D5-branes. This solution has the right structure to be dual to $\mathcal{N} = 1$ supersymmetric gauge theories with flavors, in agreement with the proposal in [10]. It has been shown that such background could be realized as integrable supercoset with Z_4 structure [9].

For $AdS_5 \times S^1$ background, the $su(2,2|2)$ algebra and its Z_4 structure were studied in [9]. Here we redefine the generator as

$$\begin{aligned}
D &= M_{45}, \qquad P_a = M_{a5} - M_{a4}, \qquad K_a = M_{a5} + M_{a4},\\
Q^{\alpha\alpha'} &= \frac{1}{2}\varepsilon^{\alpha\beta}C^{\alpha'\beta'}(Q^1_{\beta\beta'} - iQ^2_{\beta\beta'}), \qquad \bar{Q}^{\dot{\alpha}}_{\alpha'} = -\frac{1}{2}(Q^{1\dot{\alpha}}_{\alpha'} + iQ^{2\dot{\alpha}}_{\alpha'}),\\
S_{\alpha\alpha'} &= -\frac{1}{2}(Q^1_{\alpha\alpha'} + iQ^2_{\alpha\alpha'}), \qquad \bar{S}^{\alpha'}_{\dot{\alpha}} = \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}C^{\alpha'\beta'}(Q^{1\dot{\beta}}_{\beta'} - iQ^{2\dot{\beta}}_{\beta'}),
\end{aligned} \tag{2.1}$$

then we have the non-trivial brackets of the algebra

$$\begin{aligned}
[D, P_a] &= P_a, \qquad [D, K_a] = -K_a,\\
[P_a, K_b] &= 2\eta_{ab}D + 2M_{ab},\\
[P_a, M_{bc}] &= \eta_{ab}P_c - \eta_{ac}P_b, \qquad [K_a, M_{bc}] = \eta_{ab}K_c - \eta_{ac}K_b,\\
[M_{ab}, M_{cd}] &= \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac},\\
[D, Q^{\alpha\alpha'}] &= \frac{1}{2}Q^{\alpha\alpha'}, \qquad [D, S_{\alpha\alpha'} = -\frac{1}{2}S_{\alpha\alpha'},\\
[T_{a'}, T_{b'}] &= \varepsilon_{a'b'c'}T_{c'},\\
[P_a, S_{\alpha\alpha'}] &= -i\bar{Q}^{\dot{\alpha}}_{\alpha'}(\bar{\sigma}_a)_{\dot{\alpha}\alpha}, \qquad [K_a, Q^{\alpha\alpha'}] = -i\bar{S}^{\dot{\alpha}\alpha'}(\bar{\sigma}_a)_{\dot{\alpha}\alpha},\\
[M_{ab}, Q^{\alpha\alpha'}] &= \frac{1}{2}Q^{\beta\alpha'}(\sigma_{a\bar{b}})^{\alpha}_{\beta}, \qquad [M_{ab}, S_{\alpha\alpha'}] = \frac{1}{2}S_{\beta\alpha'}(\sigma_{\bar{a}b})^{\alpha}_{\beta},\\
[T_{a'}, Q^{\alpha\alpha'}] &= \frac{1}{2}Q^{\alpha\beta'}(\tau_{a'})^{\beta'}_{\alpha'}, \qquad [T_{a'}, S^{\alpha\alpha'}] = \frac{1}{2}S^{\alpha\beta'}(\tau_{a'})^{\beta'}_{\alpha'},\\
[Q^{\alpha\alpha'}, R] &= \frac{i}{2}Q^{\alpha\alpha'}, \qquad [S_{\alpha\alpha'}, R] = -\frac{i}{2}S_{\alpha\alpha'},\\
\{Q^{\alpha\alpha'}, \bar{Q}^{\dot{\alpha}}_{\beta'}\} &= (\sigma^a)^{\alpha\dot{\alpha}}\delta^{\alpha'}_{\beta'}P_a, \qquad \{S_{\alpha\alpha'}, \bar{S}^{\beta'}_{\dot{\alpha}}\} = (\sigma^a)_{\alpha\dot{\alpha}}\delta^{\beta'}_{\alpha'}K_a,\\
\{Q^{\alpha\alpha'}, S_{\beta\beta'}\} &= \delta^{\alpha}_{\beta}\delta^{\alpha'}_{\beta'}\left[i\left(D + \frac{1}{2}\gamma^{ab}M_{ab}\right) - R\right] - 2i\delta^{\alpha}_{\beta}(\tau^{a'})^{\alpha'}_{\beta'}T_{a'}.
\end{aligned} \tag{2.2}$$

Here $a, b = 0, 1, 2, 3$ are the $so(1,3)$ indices, $\alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = 1, 2$ are the $so(1,3)$ spinor indices, which are lowered and raised using $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon^{12} = -\epsilon^{21} = -1$, $\epsilon_{\dot{1}\dot{2}} = -\epsilon_{\dot{2}\dot{1}} = 1, \epsilon^{\dot{1}\dot{2}} = -\epsilon^{\dot{2}\dot{1}} = -1$. The matrices $(\eta_{ab}) = (\eta^{ab}) = \mathrm{diag}(- + ++)$, and the Dirac matrices are formed by $\sigma^a = (\mathbf{1}, \sigma^i)$, $\bar{\sigma}^a = (\mathbf{1}, -\sigma^i)$, $\sigma^{a\bar{b}} = \frac{1}{2}[\sigma^a, \bar{\sigma}^b]$. And $a', b' = 1, 2, 3$ are the $so(3)$ indices, $\alpha', \beta' = 1, 2$ are the $so(3)$ spinor indices, which are lowered and raised using $C_{\alpha'\beta'} = \eta_{\alpha'\beta'}$. The matrices $(\eta_{a'b'}) = \mathrm{diag}(+ + +)$, and the Dirac matrices are $\tau_{a'} = -\mathrm{i}\sigma_{a'}$. $T_{a'}$ and R are the generators of $su(2)$ and $u(1)$ respectively.

The Z_4-automorphism invariant subspaces are classified as

$$\begin{aligned}
\mathcal{H}_0 &= \{P_a - K_a, J_{ab}, T_{a'}\},\\
\mathcal{H}_1 &= \{\varepsilon_{\alpha\beta}C_{\alpha'\beta'}Q^{\beta\beta'} - S_{\alpha\alpha'}, \varepsilon^{\dot\alpha\dot\beta}C_{\alpha'\beta'}\bar{S}^{\beta'}_{\dot\beta} - \bar{Q}^{\dot\alpha}_{\alpha'}\},\\
\mathcal{H}_2 &= \{P_a + K_a, D, R\},\\
\mathcal{H}_3 &= \{\varepsilon_{\alpha\beta}C_{\alpha'\beta'}Q^{\beta\beta'} + S_{\alpha\alpha'}, \varepsilon^{\dot\alpha\dot\beta}C_{\alpha'\beta'}\bar{S}^{\beta'}_{\dot\beta} + \bar{Q}^{\dot\alpha}_{\alpha'},\}
\end{aligned} \tag{2.3}$$

where $\mathcal{H}_i$ denotes the subspace of grading i.

The non-vanishing components of the Cartan-Killing bilinear forms are

$$\begin{aligned}
&\mathrm{Str}(P_aK_b) = -2\eta_{ab}, \qquad \mathrm{Str}(DD) = 1, \qquad \mathrm{Str}(J_{ab}J_{cd}) = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc},\\
&\mathrm{Str}(RR) = 4, \qquad \mathrm{Str}(T_{a'}T_{b'}) = -\frac{1}{2}\delta_{a'b'}, \qquad \mathrm{Str}(Q^{\alpha\alpha'}S_{\beta\beta'}) = 2i\delta^{\alpha}_{\beta}\delta^{\alpha'}_{\beta'}.
\end{aligned} \tag{2.4}$$

A general group element $g \in \mathrm{SU}(2,2|2)$ can be parameterized as

$$g = \exp(x^aP_a + x'^aK_a + \theta_{\alpha\alpha'}Q^{\alpha\alpha'} + \xi^{\alpha\alpha'}S_{\alpha\alpha'})\exp(\bar\theta^{\alpha'}_{\dot\alpha}\bar{Q}^{\dot\alpha}_{\alpha'} + \bar\xi^{\dot\alpha}_{\alpha'}\ \bar{S}^{\alpha'}_{\dot\alpha})y^D\exp(R). \tag{2.5}$$

Now we use the κ-symmetry to fix $\xi^{\alpha\alpha'} = 0$, and use the gauge symmetry to set $x'^a = 0$, then we read the coset representative

$$\begin{aligned}
g &= \exp(x^aP_a + \theta_{\alpha\alpha'}Q^{\alpha\alpha'})\exp(\bar\theta^{\alpha'}_{\dot\alpha}\bar{Q}^{\dot\alpha}_{\alpha'} + \bar\xi^{\dot\alpha}_{\alpha'}\ \bar{S}^{\alpha'}_{\dot\alpha})y^D\exp(R),\\
&\equiv \exp(x^aP_a + \theta_{\alpha\alpha'}Q^{\alpha\alpha'})e^B.
\end{aligned} \tag{2.6}$$

The Green-Schwarz sigma-model on the supercosets of supergroup G with $\mathbf{Z_4}$ automorphism is generically described by the action

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\mathrm{Str}\left(J_2\bar{J}_2 + \frac{1}{2}J_1\bar{J}_3 - \frac{1}{2}J_3\bar{J}_1\right), \tag{2.7}$$

where R is the AdS radius, $J = g^{-1}\partial g$ for $g \in G$ and J_i is the current J restricted to the invariant subspace $\mathcal{H}_i$ of the $\mathbf{Z_4}$ automorphism of the algebra of the group G. In the case at hand, using the above algebra, the sigma-model (2.7) takes the form

$$\begin{aligned}
S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&-(J_{P_a}+J_{K_a})(\bar{J}_{P_b}+\bar{J}_{K_b})\eta^{ab} + J_D\bar{J}_D + 4J_R\bar{J}_R \qquad (2.8)\\
&+i\varepsilon^{\alpha\beta}C^{\alpha'\beta'}(J_{Q_{\alpha\alpha'}}\bar{J}_{Q_{\beta\beta'}} - J_{S_{\alpha\alpha'}}\bar{J}_{S_{beta\beta'}}) + i\varepsilon_{\dot\alpha\dot\beta}C^{\alpha'\beta'}(J_{\bar{Q}^{\dot\alpha}_{\alpha'}}\bar{J}_{\bar{Q}^{\dot\beta}_{\beta'}} - J_{\bar{S}^{\dot\alpha}_{\alpha'}}\bar{J}_{\bar{S}^{\dot\beta}_{\beta'}})\Big],
\end{aligned}$$

where the currents take the form

$$\begin{aligned}
J_{P_a} &= [e^{-B}(dx^aP_a + d\theta_{\alpha\alpha'}Q^{\alpha\alpha'})e^B]_{P_a}, & J_{Q^{\alpha\alpha'}} &= [e^{-B}(dx^aP_a + d\theta_{\alpha\alpha'}Q^{\alpha\alpha'})e^B]_{Q^{\alpha\alpha'}},\\
J_K &= 0, & J_{\bar{Q}^{\dot\alpha}_{\alpha'}} &= [e^{-B}de^B]_{\bar{Q}^{\dot\alpha}_{\alpha'}},\\
J_{S_{\alpha\alpha'}} &= 0, & J_{\bar{S}^{\alpha'}_{\dot\alpha}} &= [e^{-B}de^B]_{\bar{S}^{\alpha'}_{\dot\alpha}},\\
J_D &= [e^{-B}de^B]_D, & J_R &= [e^{-B}de^B]_R.
\end{aligned} \tag{2.9}$$

We can now T-dualize the action with respect to x^a and $\theta_{\alpha\alpha'}$ via Buscher's procedure [13]. By introducing the bosonic gauge fields $(A^a, \bar{A}^a)$ for the translation P_a, the fermionic gauge fields $(A_{\alpha\alpha'}, \bar{A}_{\alpha\alpha'})$ for the supercharges $Q^{\alpha\alpha'}$, and the Lagrange multipliers $\tilde{x}_a$ and $\tilde{\theta}^{\alpha\alpha'}$, adding the Lagrange multiplier term

$$\frac{R^2}{4\pi\alpha'}\int d^2z[\tilde{x}_a(\bar{\partial}A^a - \partial\bar{A}^a) + \tilde{\theta}^{\alpha\alpha'}(\bar{\partial}A_{\alpha\alpha'} - \partial\bar{A}_{\alpha\alpha'})] \tag{2.10}$$

to the action (2.8), we have the full action

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z[-\eta_{ab}A'^a\bar{A}'^b + i\varepsilon^{\alpha\beta}C^{\alpha'\beta'}A'_{\alpha\alpha'}\bar{A}'_{\beta\beta'} + \dots \\ +\tilde{x}_a(\bar{\partial}A^a - \partial\bar{A}^a) + \tilde{\theta}^{\alpha\alpha'}(\bar{\partial}A_{\alpha\alpha'} - \partial\bar{A}_{\alpha\alpha'})] \tag{2.11}$$

where ... denotes the spectator terms and

$$A'^a = [e^{-B}(A^bP_b + A_{\alpha\alpha'}Q^{\alpha\alpha'})e^B]_{P_a}, \quad A'_{\alpha\alpha'} = [e^{-B}(A^aP_a + A_{\beta\beta'}Q^{\beta\beta'})^B_e]_{Q^{\alpha\alpha'}}. \tag{2.12}$$

After plugging the inverse relations $A^a = [e^B(A'^bP_b + A'_{\alpha\alpha'}Q^{\alpha\alpha'})e^{-B}]_{P_a}$ and $A_{\alpha\alpha'} = [e^B(A'^aP_a + A'_{\beta\beta'}Q^{\beta\beta'})e^{-B}]_{Q^{\alpha\alpha'}}$ into the action, we can integrate out A'^a and $A'_{\alpha\alpha'}$ by using their equations of motion

$$\begin{aligned}
A'^a &= \eta^{ab}([e^B\partial\tilde{x}_bP_be^{-B}]_{P_b} + \partial[e^B\tilde{\theta}^{\alpha\alpha'}P_be^{-B}]_{Q^{\alpha\alpha'}}) \\
&= \eta^{ab}[e^{-B}(\partial\tilde{x}_cK_c + i\partial\tilde{\theta}^{\alpha\alpha'}S_{\alpha\alpha'})e^B]_{K_b}\,, \\
\bar{A}'^a &= -\eta^{ab}([e^B\bar{\partial}\tilde{x}_bP_be^{-B}]_{P_b} + [e^B\bar{\partial}\tilde{\theta}^{\alpha\alpha'}P_be^{-B}]_{Q^{\alpha\alpha'}}) \\
&= -\eta^{ab}[e^{-B}(\bar{\partial}\tilde{x}_cK_c + i\bar{\partial}\tilde{\theta}^{\alpha\alpha'}S_{\alpha\alpha'})e^B]_{K_\alpha}\,, \\
A'_{\alpha\alpha'} &= -i\varepsilon_{\alpha\beta}C_{\alpha'\beta'}([e^B\partial\tilde{x}_aQ_{\beta\beta'}e^{-B}]_{P_a} - [e^B\partial\tilde{\theta}^{\gamma\gamma'}Q_{\beta\beta'}e^{-B}]_{Q_{\gamma\gamma'}}) \\
&= \varepsilon_{\alpha\beta}C_{\alpha'\beta'}[e^{-B}(\partial\tilde{x}_aK_a + i\partial\tilde{\theta}^{\gamma\gamma'}S_{\gamma\gamma'})e^B]_{S_{\beta\beta'}}, \\
\bar{A}'_{\alpha\alpha'} &= i\varepsilon_{\alpha\beta}C_{\alpha'\beta'}([e^B\bar{\partial}\tilde{x}_aQ_{\beta\beta'}e^{-B}]_{P_a} - [e^B\bar{\partial}\tilde{\theta}^{\gamma\gamma'}Q_{B\beta'}e^{-B}]_{Q_{\gamma\gamma'}}) \\
&= -\varepsilon_{\alpha\beta}C_{\alpha'\beta'}[e^{-B}(\bar{\partial}\tilde{x}_aK_a + i\bar{\partial}\tilde{\theta}^{\gamma\gamma'}S_{\gamma\gamma'})e^B]_{S_{\beta\beta'}}.
\end{aligned} \tag{2.13}$$

Finally we obtain the T-dualized action

$$\begin{aligned}
S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&-[e^{-B}(\partial\tilde{x}_cK_c + i\partial\tilde{\theta}^{\alpha\alpha'}S_{\alpha\alpha'})e^B]_{K_a}[e^{-B}(\partial\tilde{x}_cK_c + i\partial\tilde{\theta}^{\alpha\alpha'}S_{\alpha\alpha'})e^B]_{K_b}\eta^{ab} \\
&-i\varepsilon^{\alpha\beta}C^{\alpha'\beta'}[e^{-B}(\bar{\partial}\tilde{x}_aK_a + i\bar{\partial}\tilde{\theta}^{\gamma\gamma'}S_{\gamma\gamma'})e^B]_{S_{\alpha\alpha'}}[e^{-B}(\bar{\partial}\tilde{x}_aK_a + i\bar{\partial}\tilde{\theta}^{\gamma\gamma'}S_{\gamma\gamma'})e^B]_{S_{\beta\beta'}} \\
&+\dots\Big]
\end{aligned} \tag{2.14}$$

Note that the $su(2,2|2)$ algebra admits the automorphism

$$P_a \leftrightarrow K_a\,, \quad D \to -D\,, \quad Q^{\alpha\alpha'} \leftrightarrow S_{\alpha\alpha'}\,, \quad \bar{Q}^{\dot{\alpha}}_{\alpha'} \leftrightarrow \bar{S}^{\alpha'}_{\dot{\alpha}}\,, \tag{2.15}$$

with the rest of the generators unchanged. Applying this automorphism combined with the change of variables

$$x \to \tilde{x}\,, \quad \theta_{\alpha\alpha'} \to i\tilde{\theta}^{\alpha\alpha'}\,, \quad \bar{\theta}^{\alpha'}_{\dot{\alpha}} \leftrightarrow \bar{\xi}^{\dot{\alpha}}_{\alpha'}\,, \quad y \to y\,, \tag{2.16}$$

to (2.8), we recover (2.14). This completes our proof that the background $AdS_5 \times S^1$ is self-dual under fermionic T-duality.

JHEP12(2009)051

3 $AdS_2 \times CP^n$ background

In this section, we turn to the sigma models on the $AdS_2 \times CP^n$ backgrounds. We restrict ourselves to the critical and noncritical superstrings with $n \neq 4$. For $n = 1$, since CP^1 is just two-dimensional sphere S^2, we have $AdS_2 \times S^2$, which has been studied in [7]. The superstring propagating in the $\mathrm{AdS_2 \times CP^n}$ background has the bosonic part

$$\mathrm{AdS_2 \times CP^n \cong SO(1,2)/U(1) \times SU(n+1)/U(n)}. \tag{3.1}$$

The supergroups which have bosonic subgroups $\mathrm{SO(1,2) \times SU(n+1)}$ can be $\mathrm{SU(1,1|n+1)}$ for $n = 1,2,3,4$, and $\mathrm{OSp(3|2)}$ for $n = 1$ and $\mathrm{OSp(6|2)}$ for $n = 3$. This means that for $n = 1,3$, we may have two different supercosets realizations, based on SU supergroups or on OSp supergroups, with different supercharges respectively.

3.1 PSU supergroup case

In this subsection, we will focus on the $\mathrm{SU(1,1|n+1)}$ case, with a sigma-model on the coset space

$$\frac{\mathrm{SU(1,1|n+1)}}{\mathrm{U(1) \times U(n) \times U(1)}} \tag{3.2}$$

The last U(1) is the overall generator. The super-Lie algebras $su(1,1|n)$ are the algebras of $(2+n)\times(2+n)$ matrices with bosonic diagonal blocks and fermionic off-diagonal blocks

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \quad \text{with } \ trA = trB = 0, \tag{3.3}$$

where A and B are even(bosonic) 2×2 and $n \times n$ matrices. The $2 \times n$ matrix X and $n \times 2$ matrix Y are odd. The anti-hermiticity condition is

$$M^\dagger \equiv \begin{pmatrix} \sigma_3 A^\dagger \sigma_3 & -i\sigma_3 Y^\dagger \\ -iX^\dagger \sigma_3 & B^\dagger \end{pmatrix} = -M, \tag{3.4}$$

which leads to

$$A = -\sigma_3 A^\dagger \sigma_3^{-1}\,, \quad B = -B^\dagger\,, \quad X = i\sigma_3 Y^\dagger. \tag{3.5}$$

The algebra $su(1,1|n)$ has a $\mathbf{Z}_4$ automorphism, generated by the conjugation map $M \to \Omega(M) \equiv \Omega M \Omega^{-1}$ with the matrix

$$\Omega = \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & i\mathbf{I}_{n-1} & 0 \\ 0 & 0 & -i \end{pmatrix}\,. \tag{3.6}$$

This conjugation respects the anti-hermiticity conditions given above and manifestly gives an algebra automorphism, $\Omega^4(M) = M$. In addition, the invariant subalgebra $\Omega(M) = M$ is the desired bosonic $u(1) \oplus u(n-1) \oplus u(1)$ algebra.

Using this automorphism the algebra can be decomposed into $\mathbf{Z}_4$-invariant subspaces $\mathcal{H}_k$ $(k = 0\ldots,3)$ such that

$$\mathcal{H}_k = \{X \in \mathrm{su}(1,1|n)|\Omega X \Omega^{-1} = i^k X\} \ . \tag{3.7}$$

The $su(1,1|n)$ algebra is generated by the following (anti)commutators:

$$\begin{aligned}
&[D,P] = P, && [D,K] = -K,\\
&[P,K] = -2D, && [R_i^j, R_k^l] = \delta_k^j R_i^l - \delta_i^l R_k^j,\\
&[D,Q^i] = \frac{1}{2}Q^i, && [D,S_i] = -\frac{1}{2}S_i,\\
&[P,Q^i] = 0, && [P,S_i] = i\bar{Q}_i,\\
&[K,Q^i] = i\bar{S}^i, && [K,S_i] = 0,\\
&[R_i^j, Q^k] = -(\delta_i^k Q^j - \frac{1}{n}\delta_i^j Q^k), && [R_i^j, S_k] = \left(\delta_k^j S_i - \frac{1}{n}\delta_i^j S_k\right),\\
&\{Q^i, \bar{Q}_j\} = \delta_j^i P, && \{S_i, \bar{S}^j\} = -\delta_j^i K,\\
&\{Q^i, S_j\} = -\mathrm{i}(\delta_j^i(A+D) + R_j^i), && \\
&[Q^i, A] = -\frac{n-2}{n}Q^i, && [S_i, A] = \frac{n-2}{n}S_i,
\end{aligned} \tag{3.8}$$

where $i,j = 1,2,\ldots,n$ are SU(n) R-symmetry indices, and A is the overall U(1) generator

$$A = \left(\begin{array}{cc|cccc}
1 & 0 & 0 & 0 & \ldots & 0\\
0 & 1 & 0 & 0 & \ldots & 0\\
\hline
0 & 0 & \frac{2}{n} & 0 & \ldots & 0\\
0 & 0 & 0 & \frac{2}{n} & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \ldots & \frac{2}{n}
\end{array}\right) . \tag{3.9}$$

The definition of other generators could be found in appendix A. Notice we neglect the AdS_2 spinor index $\alpha = 1$ of the fermionic generators (eg. $Q^{\alpha i}$).

The $\mathbf{Z_4}$-graded subspaces of the algebra are

$$\begin{aligned}
\mathcal{H}_0 &= \{P+K, R_a^b, A\},\\
\mathcal{H}_1 &= \{Q^a - \bar{S}^a, Q^d + \bar{S}^d, \bar{Q}_a - S_a, \bar{Q}_d + S_d\},\\
\mathcal{H}_2 &= \{P-K, D, R_n^a, R_a^n\},\\
\mathcal{H}_3 &= \{Q^a + \bar{S}^a, Q^d - \bar{S}^d, \bar{Q}_a + S_a, \bar{Q}_d - S_d\},
\end{aligned} \tag{3.10}$$

where $a,b = 1,2,\cdots,n-1$. The non-vanishing components of the Cartan-Killing bilinear form are

$$\begin{aligned}
&\mathrm{Str}(PK) = -1, && \mathrm{Str}(DD) = \frac{1}{2},\\
&\mathrm{Str}(R_i^j R_k^l) = -\left(\delta_i^l \delta_k^j - \frac{1}{n}\delta_i^j \delta_k^l\right), && \\
&\mathrm{Str}(Q^i S_j) = -\frac{i}{2}, && \mathrm{Str}(\bar{Q}_i \bar{S}^j) = \frac{i}{2}.
\end{aligned} \tag{3.11}$$

In order to study the fermionic T-duality, it turns out to be convenient to redefine the generators. Instead of using the above algebra directly, we redefine the Grassmann-odd generators as linear combinations of the original ones

$$
\begin{aligned}
Q^a &= \frac{Q^a + \bar{Q}_n}{2}, & \hat{Q}_a &= \frac{Q^a - \bar{Q}_n}{2}, \\
S^a &= \frac{\bar{S}^a - S_n}{2}, & \hat{S}_a &= \frac{\bar{S}^a + S_n}{2}, \\
Q^n &= \frac{\Sigma\bar{Q}_a - Q_n}{2}, & \hat{Q}_n &= \frac{-Q^n - \Sigma\bar{Q}_n}{2}, \\
S^n &= \frac{-\Sigma S_a - \bar{S}^n}{2}, & \hat{S}_a &= \frac{\Sigma S_a - \bar{S}^n}{2},
\end{aligned}
\tag{3.12}
$$

where the sum is over $1, 2, \ldots, n-1$. The $\mathbf{Z_4}$ invariant subspaces of the algebra change to

$$
\begin{aligned}
\mathcal{H}_0 &= \{P + K, R_a^b, A\}, \\
\mathcal{H}_1 &= \{Q^a - S^a, Q^n + S^n, \hat{Q}_a - \hat{S}_a, \hat{Q}_n + \hat{S}_n\}, \\
\mathcal{H}_2 &= \{P - K, D, R_n^a, R_a^n\}, \\
\mathcal{H}_3 &= \{Q^a + S^a, Q^n - S^n, \hat{Q}_a + \hat{S}_a, \hat{Q}_n - \hat{S}_n\},
\end{aligned}
\tag{3.13}
$$

and the nonvanishing Cartan-Killing bilinear form of the fermionic generators change to

$$
\mathrm{Str}(Q^i S^j) = iC_{ij}, \quad \mathrm{Str}(\hat{Q}_i \hat{S}_j) = -iC_{ij}, \tag{3.14}
$$

where

$$
C_{ij} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -1 & -1 & \ldots & -1 & 0 \end{pmatrix}.
$$

The sigma-model action (2.7) now takes the form

$$
\begin{aligned}
S = \frac{R^2}{4\pi i'} \int d^2z \bigg[& \frac{1}{2}(J_P - J_K)(\bar{J}_P - \bar{J}_K) + \frac{1}{2} J_D \bar{J}_D + \frac{1}{2} J_{R_a^n} \bar{J}_{R_n^a} - \\
& - \frac{i}{2} \eta_{ij} (J_{Q_i} \bar{J}_{Q_j} - J_{\hat{Q}_i} \bar{J}_{\hat{Q}_j} + J_{S_i} \bar{J}_{S_j} - J_{\hat{S}_i} \bar{J}_{\hat{S}_j}) \bigg],
\end{aligned}
\tag{3.15}
$$

where $\eta_{an} = \eta_{na} = 1$ and zero otherwise. This is exactly the same as the one in [7] if we take $n = 2$.

Next, after fixing the kappa symmetry and the gauge symmetry, we parameterize the coset element as

$$
g = e^{xP + \theta_i Q^i} e^B, \tag{3.16}
$$

where

$$
e^B \equiv e^{\hat{\theta}^i \hat{Q}_i + \hat{\xi}^i \hat{S}_i} y^D e^{\Sigma y_j^i R_i^j / y}. \tag{3.17}
$$

JHEP12(2009)051

The components of the Maurer-Cartan 1-form are

$$\begin{aligned}
J_P &= [e^{-B}(dxP + d\theta_i Q^i)e^B]_P\,, & J_{Q^i} &= [e^{-B}(dxP + d\theta_i Q^i)e^B]_{Q^i}\,,\\
J_K &= 0\,, & J_{\hat{Q}_i} &= [e^{-B}de^B]_{\hat{Q}_i}\,,\\
J_{S^i} &= 0\,, & J_{\hat{S}_i} &= [e^{-B}de^B]_{\hat{S}_i}\,,\\
J_D &= [e^{-B}de^B]_D\,, & J_{R_i^j} &= [e^{-B}de^B]_{R_i^j}\,. \qquad (3.18)
\end{aligned}$$

We would like to do T-dual transformation in the directions of the Abelian sub-algebra formed by the generators P and Q^i. Similar to the case in section 2, we introduce the bosonic gauge fields $(A, \bar{A})$ for the translation P and the fermionic gauge fields $(A_i, \bar{A}_i)$ for the supercharges Q^i, and add the Lagrange multiplier term

$$\frac{R^2}{4\pi\alpha'}\int d^2z[\tilde{x}(\bar{\partial}A - \partial\bar{A}) + \tilde{\theta}^i(\bar{\partial}A_i - \partial\bar{A}_i)] \qquad (3.19)$$

with $\tilde{x}$ and $\tilde{\theta}^i$ being multiplier, to the action (3.15). Then the full action takes the form

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[\frac{1}{2}A'\bar{A}' - \frac{i}{2}\eta^{ij}A'_i\bar{A}'_j + \dots + \tilde{x}(\bar{\partial}A - \partial\bar{A}) + \tilde{\theta}^i(\bar{\partial}A_i - \partial\bar{A}_i)\Big], \qquad (3.20)$$

where

$$A' = [e^{-B}(AP + A'_iQ^i)e^B]_P\,, \quad A'_i = [e^{-B}(AP + A'_iQ^i)e^B]_{Q^i}. \qquad (3.21)$$

With the inverse $A = [e^B(dxP + d\theta_iQ^i)e^{-B}]_P$ and $A_i = [e^B(dP + d\theta_iQ^i)e^{-B}]_{Q^i}$, we find the equations of motion

$$\begin{aligned}
A' &= -2[e^B\partial\tilde{x}Pe^{-B}]_P - 2[e^B\partial\tilde{\theta}^iPe^{-B}]_{Q^i} = -2[e^{-B}(\partial\tilde{x}K + i\partial\tilde{\theta}^iS_i)e^B]_K\,,\\
\bar{A}' &= 2[e^B\bar{\partial}\tilde{x}Pe^{-B}]_P + 2[e^B\bar{\partial}\tilde{\theta}^iPe^{-B}]_{Q^i} = 2[e^{-B}(\bar{\partial}\tilde{x}K + i\bar{\partial}\tilde{\theta}^iS_i)e^B]_K\,,\\
A'_i &= 2i\eta_{ij}([e^B\partial\tilde{x}Q^je^{-B}]_P - [e^B\partial\tilde{\theta}^kQ^je^{-B}]_{Q^k}) = -2\eta_{ij}[e^{-B}(\partial\tilde{x}K + i\partial\tilde{\theta}^kS_k)e^B]_{S_j}\,,\\
\bar{A}'_i &= 2i\eta_{ij}([e^B\bar{\partial}\tilde{x}Q^je^{-B}]_P - [e^B\bar{\partial}\tilde{\theta}^kQ^je^{-B}]_{Q^k}) = -2\eta_{ij}[e^{-B}(\bar{\partial}\tilde{x}K + i\bar{\partial}\tilde{\theta}^kS_k)e^B]_{S_j}.
\end{aligned}$$

Integrating out A' and A'^i, and rescaling $\tilde{x} \to \frac{1}{2}\tilde{x}$ and $\tilde{\theta}^i \to \frac{1}{2}\tilde{\theta}^i$, we have the T-dualized action

$$\begin{aligned}
S_T = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&\frac{1}{2}[e^{-B}(\partial\tilde{x}K + i\partial\tilde{\theta}^iS_i)e^B]_K[e^{-B}(\bar{\partial}\tilde{x}K + i\bar{\partial}\tilde{\theta}^iS_i)e^B]_K \qquad (3.22)\\
&- \frac{i}{2}\eta_{ij}[e^{-B}(\partial\tilde{x}K + i\partial\tilde{\theta}^kS_k)e^B]_{S_i}[e^{-B}(\bar{\partial}\tilde{x}K + i\bar{\partial}\tilde{\theta}^kS_k)e^B]_{S_j} + \dots\Big],
\end{aligned}$$

Using the automorphism of the algebra

$$P \leftrightarrow K\,, \quad D \to -D\,, \quad Q^i \leftrightarrow S_i\,, \quad \hat{Q}_i \leftrightarrow \hat{S}_i\,, \qquad (3.23)$$

and changing the variables

$$x \to \tilde{x}\,, \quad \theta_i \to i\tilde{\theta}^i\,, \quad \hat{\theta}_i \leftrightarrow \hat{\xi}_i\,, \quad y_i^j \to \frac{y_i^j}{y^2}\,, \qquad (3.24)$$

we find that the action (3.15) is the same as (3.22). This shows that the supercosets of $\mathrm{SU}(1,1|n)$ group is self-dual under fermionic T-duality.

3.2 The OSp case

In the above subsection we discussed the $AdS_2 \times CP^n$ supercoset models based on PSU supergroups. In this section, we will discuss another realization of $AdS_2 \times CP^n$ based on ortho-symplectic supergroups. As we have said before, there are only two cases, $OSp(3|2)$ for $n = 1$ and $OSp(6|2)$ for $n = 3$.

3.2.1 The $OSp(3|2)$ case

For n=1 case, there is another supercoset realization of $AdS_2 \times CP^1 \cong \mathrm{SO}(1,2)/\mathrm{SO}(1,1) \times \mathrm{SO}(3)/\mathrm{SO}(2)$. The supergroup $OSp(3|2)$ corresponds to the superalgebra $B(1,1)$, with its bosonic subgroup being $\mathrm{SO}(3) \times Sp(2)$. It has six real fermionic generators transforming as the $(3,2)$ representation of $\mathrm{SO}(3) \times Sp(2)$. This is different from its PSU realization. In these two different realizations, supercharges are totally different.

The algebra of $osp(3|2)$ is

$$\begin{aligned}
&[D,P] = P, \qquad [D,K] = -K, \qquad [P,K] = -2D, \\
&[D,Q_i] = \frac{1}{2}Q_i, \qquad [D,S_i] = \frac{1}{2}S_i \qquad [P,S_i] = iQ_i, \qquad [K,Q_i] = iS_i, \\
&[J_i,J_j] = i\varepsilon_{ijk}J_k, \quad [J_i,Q_j] = i\varepsilon_{ijk}Q_k, \quad [J_i,S_j] = i\varepsilon_{ijk}S_k, \\
&\{Q_i,Q_j\} = \delta_{ij}P, \qquad \{S_i,S_j\} = -\delta_{ij}K, \qquad \{Q_i,S_j\} = -i\delta_{ij}D - \frac{1}{2}\varepsilon_{ijk}J_k,
\end{aligned} \tag{3.25}$$

where $i = 1,2,3$ are the SO(3) indices, and the Z_4-automorphism invariant subspaces are

$$\begin{aligned}
\mathcal{H}_0 &= \{P-K, J_2\}, \\
\mathcal{H}_1 &= \{Q_i+S_i\}, \\
\mathcal{H}_2 &= \{P+K, D, J_1, J_3\}, \\
\mathcal{H}_3 &= \{Q_i-S_i\}.
\end{aligned} \tag{3.26}$$

The non-vanishing components of the Cartan-Killing bilinear form are

$$\begin{aligned}
&\mathrm{Str}(PK) = 1, \qquad \mathrm{Str}(DD) = -\frac{1}{2}, \\
&\mathrm{Str}(J_iJ_j) = 2\delta_{ij}, \qquad \mathrm{Str}(Q_iS_j) = i\delta_{ij}.
\end{aligned} \tag{3.27}$$

The action take the form

$$\begin{aligned}
S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&\frac{1}{2}(J_P+J_K)(\bar{J}_P+\bar{J}_K) - \frac{1}{2}J_D\bar{J}_D + 2J_{J_3}\bar{J}_{J_3} \\
&+2J_{J_1}\bar{J}_{J_1} - \frac{i}{2}(J_{Q_i}\bar{J}_{Q_i} - J_{S_i}\bar{J}_{S_i})\Big].
\end{aligned} \tag{3.28}$$

Before performing the T-duality, we would like to discuss a little bit about the κ-symmetry. Conventionally, it is expected that the κ-symmetry can remove half of the fermionic degrees of freedom. This is indeed true for supercosets of SU supergroups. However, for supercoset models of OSp supergroup, this is not the case any more. For example, superstring on $AdS_4 \times CP^3$ [15, 16] has only eight κ-symmetry degrees of freedom, rather than the "expected" number:twelve. Following the procedure in [15], we find that our sigma model has only two κ-symmetry degrees of freedom. The detail is given in appendix B.

After gauge fixing two fermionic parameters S_1 and S_2, we have the coset element

$$g = e^{xp+\theta^1 Q_1+\theta^2 Q_2} e^{\theta^3 Q_3+\xi^3 S_3} y^D e^{\frac{J_i y^i}{y}}. \tag{3.29}$$

Then with the (anti-)communication relation, the components of the Maurer-Cartan 1-form are

$$\begin{aligned}
&J_P = [e^{-B} dx P e^B]_P, \qquad J_D = [e^{-B} de^B]_D, \qquad J_{Q_i} = [e^{-B} d\theta^i Q_i e^B]_{Q_i},\\
&J_{S_3} = [e^{-B} de^B]_{S_3}, \qquad J_{Q_3} = [e^{-B} de^B]_{Q_3} + [e^{-B} dx P e^B]_{Q_3},\\
&J_{J_3} = [e^{-B} de^B]_{J_3} \qquad J_{J_1} = [e^{-B} de^B]_{J_1} + [e^{-B} d\theta^2 Q_2 e^B]_{J_1},
\end{aligned} \tag{3.30}$$

where $i = 1, 2$. Using the fact

$$\begin{aligned}
&[e^{-B} dx P e^B]_P = \frac{1}{y} dx, \qquad [e^{-B} d\theta^i Q_i e^B]_{Q_i} = \frac{1}{y^{1/2}},\\
&[e^{-B} dx P e^B]_{Q_3} = i\xi^3 dx, \qquad [e^{-B} d\theta^2 Q_2 e^B]_{J_1} = \frac{1}{2} d\theta^2 \xi^3,
\end{aligned}$$

and $[e^{-B} de^B]_{Q_3} = j_{Q_3}, [e^{-B} de^B]_{J_1} = j_{J_1}$, we can rewrite the action as

$$\begin{aligned}
S = \frac{R^2}{4\pi\alpha'} \int d^2 z \Bigg[&\frac{1}{2}\left(\frac{\partial x \bar{\partial} x}{y^2} + \bar{J}_{Q_3} \xi^3 \partial x + J_{Q_3} \xi^3 \bar{\partial} x \right)\\
&- \frac{i}{2}\left(\frac{\partial \theta^i \bar{\partial} \theta^i}{y} - 2i \bar{J}_{J_1} \partial \theta^2 \xi^3 - 2i J_{J_1} \bar{\partial} \theta^2 \xi^3 \right) + \dots \Bigg]
\end{aligned} \tag{3.31}$$

Here we can see that the bosonic part and the fermionic part are separated, and after the T-duality, we will have terms $\bar{J}_{Q_3} \xi^3 \partial \hat{x} - J_{Q_3} \xi^3 \bar{\partial} \hat{x}$, and $\bar{J}_{J_1} \partial \hat{\theta}^2 \xi^3 - J_{J_1} \bar{\partial} \hat{\theta}^2 \xi^3$. These terms can not be obtained from any automorphism of the algebra, so the sigma model is not self-dual.

Let us make a few remarks. Firstly, this sigma model can not be cataloged to the OSp case discussed in [7]. In that paper, the sigma models of OSp supergroups discussed belong to superalgebra $C(n)$ and $D(m, n)$, with fermionic generators $\{Q, \bar{Q}, S, \bar{S}\}$. In this case, the action includes

$$\eta^{IJ} (J_{Q^I} \bar{J}_{\bar{Q}^J} - J_{\bar{Q}^I} \bar{J}_{Q^J}),$$

which leads to singular fermionic quadratic terms and can not be T-dualized. In our case, we only have fermionic generators $\{Q, S\}$ (there are only six fermionic generators), and the quadratic terms

$$J_{Q^I} \bar{J}_{Q^I} - J_{S^I} \bar{J}_{S^I}$$

in the action. Obviously, the actions with these forms can be T-dualized. This discussion can easily be generalized to the $AdS_2 \times S^{2n}$ and $AdS_4 \times S^{2n}$ backgrounds, with the supercosets $OSp(2n+1|2)/(\mathrm{SO}(2n) \times \mathrm{U}(1))$ and $OSp(2n+1|4)/(\mathrm{SO}(2n) \times \mathrm{SO}(3,1))$. The

JHEP12(2009)051

algebras of these supergroups belong to $B(n,1)$ and $B(n,2)$ types, which also have the fermionic generators $\{Q,S\}$ and the similar regular fermionic quadratic terms.

Secondly, this action will only have regular quadratic term and be self-dual if we can gauge away half of the fermions, i.e. three superconformal charges S's. But the terms linear in J appear as there are only two κ-symmetry degrees of freedom. Such kind of term forbids the model from being self-dual. The existence of such term does not depend on the gauge choice. To see this, instead of choosing S_1 and S_2, let us gauge away Q_3 and S_3. However, even with this more symmetric gauge choice, we still get terms like $\bar{J}_D\xi^i\partial\theta^i + J_D\xi^i\bar{\partial}\theta^i$, which keeps the model from being self-dual. The similar arguments also apply to $AdS_2 \times S^{2n}$ and $AdS_4 \times S^{2n}$ backgrounds in which cases κ-symmetry can only gauged away two and four fermionic degrees of freedom respectively.

Finally, there is another subtlety regarding the number of κ-symmetries in OSp super-coset models. It was observed in [15] that for the sigma model on $AdS_4 \times CP^3$ when the string moves entirely in AdS_4, the κ-symmetry parameter ϵ vanishes so that the number of κ-symmetries increases from eight to twelve, which allows us to gauge away half of the fermions. However, this does not mean that the model is self-dual in this case since the model cannot describe the complete superstring in $AdS_4 \times CP^3$ background. Actually, it was pointed out in [17, 18] that when the superstring moves entirely in AdS_4 the classical integrability of the model is still an open issue since the model could not be taken simply as a supercoset anymore. For example, the string can move in a subspace which includes AdS_4 but is a twisted superspace rather than a supercoset. In this case, the usual analysis of integrability of supercoset model could not be applied and it is not clear if the model is still integrable or not [17]. Nevertheless, if just focused on the bosonic model, the string in AdS_4 is always integrable, as is well-known. The same issue may happen in the OSp supercoset models studied in this paper.[1]

3.2.2 The $OSp(6|2)$ case

For the n=3 case, the supercoset can be $AdS_2 \times CP^3 \cong \mathrm{SO}(1,2)/\mathrm{SO}(1,1) \times \mathrm{SO}(6)/\mathrm{U}(3)$. The supergroup $OSp(6|2)$ corresponds to the superalgebra $D(3,1)$, with bosonic subgroup $\mathrm{SO}(6) \times Sp(2)$, and it has twelve real fermionic generators transforming as in the $(6,2)$ representation of $\mathrm{SO}(6) \times Sp(2)$. It is easy to see that the algebra is similar to the one of $OSp(6|4)$ [7], which corresponds to superalgebra $D(3,2)$. We can change the $Sp(4)$ generators [7] with $Sp(2)$ generators and neglect the $\mathrm{SO}(3,1)$ spinors indices to get the algebra

$$
\begin{aligned}
&[\lambda_{k\dot{l}}, \lambda_{m\dot{n}}] = 2i(\delta_{m\dot{l}}\lambda_{k\dot{n}} - \delta_{k\dot{n}}\lambda_{m\dot{l}}), && [\lambda_{k\dot{l}}, R_{mn}] = 2i(\delta_{m\dot{l}}R_{kn} - \delta_{n\dot{l}}R_{km}),\\
&[R_{mn}, R_{kl}] = 0, && [R_{mn}, R_{\dot{k}\dot{l}}] = \frac{i}{2}(\delta_{m\dot{k}}\lambda_{n\dot{l}} - \delta_{m\dot{l}}\lambda_{n\dot{k}} - \delta_{n\dot{k}}\lambda_{m\dot{l}} + \delta_{n\dot{l}}\lambda_{m\dot{k}}),\\
&[D,P] = P, && [D,K] = -K,\\
&[P,K] = -2D, && \\
&[D,Q^l] = \frac{1}{2}Q^l, && [D,S^l] = -\frac{1}{2}S^l,\\
&[P,Q^l] = 0, && [K,S^l] = 0,
\end{aligned}
$$

[1] We would like to thank D. Sorokin for clarifications on this issue.

$$
\begin{aligned}
&[P, S^l] = -iQ^l, && [K, Q^l] = iS^l, \\
&[R_{kl}, Q^{\dot p}] = i(\delta^{\dot p l} Q^k - \delta^{\dot p k} Q^l), && [R_{kl}, S^{\dot p}] = -i(\delta^{\dot p l} S^k - \delta^{\dot p k} S^l), \\
&[R_{\dot k \dot l}, Q^p] = -i(\delta^{p\dot l} Q^{\dot k} - \delta^{p\dot k} Q^{\dot l}), && [R_{\dot k \dot l}, S^p] = i(\delta^{p\dot l} S^{\dot k} - \delta^{p\dot k} S^{\dot l}), \\
&[\lambda_{k\dot l}, Q^p] = 2i\delta^{p\dot l} Q^k, && [\lambda_{k\dot l}, S^p] = 2i\delta^{p\dot l} S^k, \\
&[\lambda_{k\dot l}, Q^{\dot p}] = -2i\delta^{\dot p k} Q^{\dot l}, && [\lambda_{k\dot l}, S^{\dot p}] = -2i\delta^{\dot p k} S^{\dot l}, \\
&\{Q^l, Q^k\} = 0, && \{Q^l, Q^{\dot k}\} = -\delta^{l\dot k} P, \\
&\{S^l, S^k\} = 0, && \{S^l, S^{\dot k}\} = -\delta^{l\dot k} K, \\
&\{Q^l, S^k\} = -R_{lk}, && \{Q^{\dot l}, S^{\dot k}\} = -R_{\dot l \dot k}, \\
&\{Q^l, S^{\dot k}\} = -i\delta^{l\dot k} D + \frac{1}{2}\lambda_{l\dot k} && \{Q^{\dot l}, S^k\} = i\delta^{\dot l k} D + \frac{1}{2}\lambda_{k\dot l},
\end{aligned} \tag{3.32}
$$

where $k, l = 1, 2, 3$ and the dotted ones are the 3 and $\bar{3}$ of $u(3)$ respectively. Note that $\lambda_{k\dot l}$'s are the generators of $so(6)$.

The algebra admits the Z_4 automorphism and the invariant subspaces are

$$
\begin{aligned}
\mathcal{H}_0 &= \{P - K, \lambda_{l\dot k}\}, \\
\mathcal{H}_1 &= \{Q^l - S^l, Q^{\dot l} - S^{\dot l}\}, \\
\mathcal{H}_2 &= \{P + K, D, R_{kl}, R_{\dot k \dot l}\}, \\
\mathcal{H}_3 &= \{Q^l + S^l, Q^{\dot l} + S^{\dot L}\}.
\end{aligned} \tag{3.33}
$$

Similar to the $OSp(6|4)$ case, it does not have a fermionic T-duality symmetry because the matrix multiplying the gauge field is singular.

4 Conclusion and discussion

We have shown that the sigma models on $AdS_5 \times S^1$ and $AdS_2 \times CP^n$ background realized as supercosets of PSU supergroups are self-dual under the combination of bosonic and fermionic T-duality, while the $AdS_2 \times CP^n$ background as the supercosets of OSp $(n = 1, 3)$ supergroups are not. For $n = 3$, the OSp sigma model is quite similar to the one on $AdS_4 \times CP^3$, in which case there is no appropriate fermionic quadratic term to do T-dualization. However, for $n = 1$ the $OSp(3|2)$ model in our case is very different from the OSp case discussed in [7]. This $OSp(3|2)$ model has appropriate fermionic quadratic term, which allows us to perform fermionic T-duality. Nevertheless, the model is not self-dual under T-duality, as there are not enough κ-symmetry degrees of freedom.

The difference between these OSp cases stems from the fact that they belong to different superalgebras. The cases in [7] belong to superalgebras $C(n)$ and $D(m, n)$, the $OSp(3|2)$ case belongs to $B(1, 1)$. For the former case, there is no appropriate fermionic quadratic terms, while for the latter case, the fermionic quadratic terms are not singular but now the κ-symmetry degrees of freedom are not enough to gauge away the right number of fermions to allow the model to be self-dual. This discussion can be generalized to $AdS_2 \times S^{2n}$ and $AdS_4 \times S^{2n}$ backgrounds, both of which could be realized as the supercosets of OSp supergroups with $B(m, n)$ type superalgebra.

Another lesson from our study is that for some coset models, they may have different supersymmetrized coset realizations, which have different behaviors under fermionic T-duality. A typical example is $AdS_2 \times S^2$. This indicates that in the study of these backgrounds, we need not only care about the bosonic backgrounds , but also need to consider the background RR-flux and the corresponding supersymmetries.

When the superstring moves only in a subspace of the supercoset, the number of κ-symmetries may be enhanced. In other words, the number of physical fermionic degrees of freedom depends on the motion of the string. In this case, it would be interesting to study the classical integrability and the self-dual properties of the model.

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A The definition of the $su(1,1|n)$ generators

The generators of the algebra can be taken as

$$
\begin{aligned}
D &= \frac{1}{2}\left(\begin{array}{cc|c} 0 & 1 & 0_{n\times 1} \\ 1 & 0 & 0_{n\times 1} \\ \hline 0_{1\times n} & 0_{1\times n} & 0_{n\times n} \end{array}\right), \\
P &= \frac{1}{2}\left(\begin{array}{cc|c} i & -i & 0_{n\times 1} \\ 1 & 0 & 0_{n\times 1} \\ \hline 0_{1\times n} & 0_{1\times n} & 0_{n\times n} \end{array}\right), \\
K &= \frac{1}{2}\left(\begin{array}{cc|c} i & i & 0_{n\times 1} \\ 1 & 0 & 0_{n\times 1} \\ \hline 0_{1\times n} & 0_{1\times n} & 0_{n\times n} \end{array}\right), \\
R_i^j &= E_{i+2,j+2} - \delta_j^i \frac{1}{n} \Sigma E_{i+2i+2}, \\
Q^i &= \frac{1}{\sqrt{2}}(E_{1,i+2} + E_{2,i+2}), \qquad \bar{S}^i = \frac{1}{\sqrt{2}}(E_{1,i+2} - E_{2,i+2}), \\
\bar{Q}_i &= \frac{i}{\sqrt{2}}(E_{i+2,1} - E_{i+2,2}), \qquad S_i = -\frac{i}{\sqrt{2}}(E_{i+2,1} + E_{i+2,2}),
\end{aligned}
\tag{A.1}
$$

where $i, j = 1, 2 \ldots n$, and

$$
E_{i,j} = \begin{cases} 1, & \text{at the } i\text{th line and } j\text{th row} \\ 0, & \text{otherwise.} \end{cases} \tag{A.2}
$$

B κ-symmetry

In this section we would like to discuss the κ-symmetry of the sigma models on $AdS_2 \times S^{2m}$ $(m = 1, 2, 3, 4)$ and $AdS_4 \times S^{2m}$, $(m = 1, 2, 3)$ backgrounds. The coset spaces for these two case are $OSp(2m + 1|2)/(\mathrm{SO}(2m) \times \mathrm{U}(1))$ and $OSp(2m + 1|4)/(\mathrm{SO}(2m) \times \mathrm{SO}(3, 1))$.

The algebra of $osp(2m + 1|2n)$ can be realized by supermatrices of the form

$$A = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix} \tag{B.1}$$

with the condition

$$X^t = -X, \quad Y^t = -C_{2n} Y C_{2n}^{1}, \quad \eta = -C_{2n}\theta^t,$$

where X and Y are even $(2m + 1) \times (2m + 1)$ and $2n \times 2n$ matrices respectively, and θ and η are the odd $(2m + 1) \times 2n$ matrix and $2n \times (2m + 1)$ matrix respectively. The matrices C_{2n} for $n = 1, 2$ are

$$C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

To get the $\mathrm{SO}(3, 1)$ (or $\mathrm{SO}(2, 1)$) part of $Sp(4)$ (or $Sp(2)$), an reality condition should be imposed.

These algebras have inner automorphism $\Omega(A) = \Omega A \Omega^{-1}$, where

$$\Omega = \begin{pmatrix} I_{2m} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix}$$

for $n = 1$ and

$$\Omega = \begin{pmatrix} I_{2m} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & C_4 \end{pmatrix}$$

for $n = 2$. The Z_4-graded subspaces are defined by

$$\mathcal{H}_k = \{X \in \mathrm{osp}(2m + 1|2n) | \Omega X \Omega^{-1} = i^k X\}.$$

The cosets $AdS_{2n} \times S^{2m}$ can be parameterized by the generators belonging to $\mathcal{H}_2$. Thus a Lie algebra element parameterizing these cosets can be presented in the form

$$A = \begin{pmatrix} y_i T_i & 0 \\ 0 & x_\mu T^\mu \end{pmatrix}$$

where $T_i = E_{i,2m+1} - E_{2m+1,i}, i = 1, 2, \ldots, 2m$,

$$T^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

JHEP12(2009)051

for $n=1$ and

$$T^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

for $n=2$.

As proved in [15], the κ-symmetry of the coset models can be understood as the local fermionic symmetry with transformation parameters $\varepsilon^{(1)}$ and $\varepsilon^{(3)}$. $\varepsilon^{(1)}$ takes the form

$$\varepsilon^{(1)} = A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\kappa^{\alpha\beta}_{++} + A^{(2)}_{\alpha,-}\kappa^{\alpha\beta}_{++}A^{(2)}_{\beta,-} + \kappa^{\alpha\beta}_{++}A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-} - \frac{1}{2n}\mathrm{str}(\Sigma A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-})\kappa^{\alpha\beta}_{++}, \quad \text{(B.2)}$$

where α, β are the world sheet indices, $A^{(2)}_{\alpha}$ is the current restrict to $\mathcal{H}_2$, $\kappa^{\alpha\beta} \in \mathcal{H}_1$ is the κ-symmetry parameter which is assumed to be independent on the dynamical fields of these models. The subscript $\pm$ in the above relation denotes the components are defined with respect to the projections defined by $V^{\alpha}_{\pm} = \frac{1}{2}(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta})V_\beta$ with $\gamma^{\alpha\beta}$ being the Weyl-invariant world-sheet metric. $\varepsilon^{(3)}$ takes a similar form. It is essential that the action remains invariant under these transformations without using the equations of motion. Thus the κ-symmetry degrees of freedom depend on the rank of ε.

Without loss of generality we can assume that the transversal fluctuations are all suppressed and the corresponding element $A^{(2)}$ has the form

$$A^{(2)} = \begin{pmatrix} yT_0 & 0 \\ 0 & ixT^0 \end{pmatrix}$$

where T^0 corresponds to time direction in the AdS Space and any element from the tangent space to S^{2m} can be brought to T_0 by SO$(2m)$ transformation. Notice that the Virasoro constraint $\mathrm{str}(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}) = 0$ demands $x^2 = y^2$ for $n=1$ and $2x^2 = y^2$ for $n=2$. Plugging this together with a generic parameter κ into eq. (B.2), we find that the ε depends on only 2 (for $n=1$) or 4 (for $n=2$) independent complex fermionic parameters. The reality condition reduces this number by half. Thus, the κ-symmetry transformation depends on 2 or 4 real fermions. Consequently the same number of fermionic degrees of freedom can be gauged away.

We have used the above method to discuss the κ-symmetry of the supercosets of SU supergroups and recovered the well-known result in thess cases successfully.

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JHEP12(2009)051

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JHEP12(2009)051